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CANONICAL FORMS FOR OPERATION TABLES OF FINITE CONNECTED QUANDLES

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Abstract. We introduce a notion of natural orderings of elements of finite connected quandles of order n. When the elements of such a quandle Q are already ordered naturally, any automophism on Q is a natural ordering. Although there are many natural orderings, the operation tables for such orderings coincide when the permutation $*q$ is a cycle of length $n-1$. This leads to the classification of automorphisms on such a quandle. Moreover, it is also shown that every row and column of the operation table of such a quandle contains all the elements of Q, which is due to K. Oshiro. We also consider the general case of finite connected quandles.

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1. Introduction

The algebraic structure of quandle was introduced by D. Joyce and S. V. Matveev in [5] and [8]. There was defined an invariant for a classical knot called knot quandle, which classifies knots up to homeomorphism of pairs.

A *quandle* is a set Q with a binary operation $* : Q \times Q \to Q$ satisfying the three axioms

(1) for any $a \in Q$, $a * a = a$,
(2) for any pair $a, b \in Q$, there exists a unique $c \in Q$ such that $a = c * b$, and
(3) for any triple $a, b, c \in Q$, we have $(a * b) * c = (a * c) * (a * c)$.

Note that possibly $a * b \neq b * a$ and $a * (b * c) \neq (a * b) * (a * c)$ for some $a, b, c \in Q$. Axiom (3) is called right-distributivity. Axiom (2) is called right-invertibility, and implies that the map $r_b : Q \ni x \mapsto x * b \in Q$ is a bijection for all $b \in Q$. A quandle is called *trivial* if r_b is the identity map for all $b \in Q$.

There is an inverse map r_b^{-1}, and we denote $r_b^{-1}(a)$ by $a \bar{*} b$. Then $\bar{*}$ gives a binary operation on Q, and under this operation Q forms a quandle $(Q, \bar{*})$, which we call the *dual* quandle of $(Q, *)$. The three formulae below are well-known.

(4) For any triple $a, b, c \in Q$, $(a * b) \bar{*} c = (a \bar{*} c) * (b \bar{*} c)$.
(5) For any triple $a, b, c \in Q$, $(a \bar{*} b) * c = (a * c) \bar{*} (b * c)$.

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	1	2	3	4	5	6
1	1	1	5	6	3	4
2	2	2	6	5	4	3
3	5	6	3	3	1	2
4	6	5	4	4	2	1
5	3	4	1	2	5	5
6	4	3	2	1	6	6

TABLE 1. An operation table for the quandle Q_{61}

(6) For any triple $a, b, c \in Q$, $a * (b * c) = (((a \bar{*} c) * b) * c)$.

For example, any group G forms a quandle under conjugation, i.e., the operation $*$ defined by $a * b = b^{-1}ab$. Such a quandle is called the *conjugation quandle* of G.

Let Q_1, Q_2 be quandles. A map $f : Q_1 \to Q_2$ is said to be a *homomorphism* if $f(a * b) = f(a) *' f(b)$ holds for all $a, b \in Q_1$, where $*$ and $*'$ are quandle operations in Q_1 and Q_2 respectively. If such a map f is bijective, then it is called an *isomorphism*, and we say that Q_1 and Q_2 are *isomorphic*. An isomorphism from a quandle Q to Q itself is called an *automorphism* of Q. Axiom (3) implies that the above bijection $r_b : Q \ni x \mapsto x * b \in Q$ is an automorphism of Q for any $b \in Q$. Actually, we can rewrite (3) as $r_c(a * b) = (a * b) * c = (a * c) * (a * c) = r_c(a) * r_c(b)$.

By distinguishing quandles, we can distinguish knots. However, knot quandles have infinite number of elements, and it is a hard problem to decide given two knot quandles are isomorphic or not. Considering homomorphism from a knot quandle to a finite quandle gives a convenient way to distinguish knot quandles. See, for example, [3], [1], [6] and [2]. For example, the involutory knot quandle is the knot quandle Q with the condition $(a * b) * b = a$ for any pair $a, b \in Q$ added, and is finite for many knots. See Section 19 in [5].

A quandle Q is said to be connected (or indecomposable) if the orbit
$O(a) = \{(\cdots((a \star_1 b_1) \star_2 b_2) \star_3 \cdots \star_m b_m) \mid b_i \in Q,\ \star_i \in \{*, \bar{*}\},\ m \in \{0, 1, 2, \cdots\}\}$
is equal to Q for all $a \in Q$. Knot quandles are known to be connected. L. Vendramin classified connected quandles with 35 or smaller number of elements in [11].

Let Q be a finite quandle of order n, with its n elements ordered, say, $q_1, q_2, \cdots, q_n$. We write them $1, 2, \cdots, n$ for short. The *quandle matrix* for Q is an $n \times n$ matrix whose element q_{ij} in row i and column j is $i * j$. Note that the diagonal element q_{ii} is equal to i by Axiom (1). The operation table for the quandle Q_{61} in Vendramin's list is displayed in Table 1. In the right bottom 6×6 quandle matrix for Q_{61}, we can see $3 * 4 = 3$ from the element in row 3 and column 4, for example. There are some papers on quandle matrices. See, for example, [4], [10] and [9]. Let $\nu : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ be a bijection.

After reordering the elements of Q by ν, the new quandle matrix has the element $\nu(i*j)$ in row $\nu(i)$ and column $\nu(j)$ (see Definition 1 in [4]), and ν is an automorphism of Q if and only if the quandle matrix is unchanged under ν (Corollary 5 in [4]). For example, the automorphism r_b does not change the quandle matrix for any $b \in Q$.

Since the map $r_j : Q \ni x \mapsto x * j \in Q$ is a bijection, it can be regarded as a permutation on the set $\{1, 2, \cdots, n\}$. We can see the permutation r_j in the jth column of the quandle matrix for Q. For instance, the 4th column in the quandle matrix in Table 1 shows that the operation $*4$ gives a permutation $r_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 3 & 4 & 2 & 1 \end{pmatrix} = (1\ 6)(2\ 5)(3)(4)$, where $(1\ 6)$, $(2\ 5)$, (3) and (4) are mutually disjoint cycles.

As is well-known, any permutation σ is decomposed into a product of disjoint cycles uniquely modulo ordering of the cycles. When $\sigma = (i_{1,1}\ \cdots\ i_{1,\ell_1})(i_{2,1}\ \cdots\ i_{2,\ell_2})\cdots(i_{k,1}\ \cdots\ i_{k,\ell_k})$, we call the multiple set of the length of the cycles $\{\ell_1, \ell_2, \cdots, \ell_k\}$ the *pattern* of σ, where a multiple set admits repeats and disregards ordering of its elements. For example, the pattern of the above permutation r_4 is $\{1, 1, 2, 2\}$. Note that the patterns of two permutations σ and ρ coincide if and only if σ and ρ are mutually conjugate, i.e., there is some permutation ω with $\rho = \omega^{-1}\sigma\omega$.

In [7], P. Lopes and D. Roseman defined the *profile* of a quandle with n elements to be the sequence of the patterns of $r_1, r_2, \cdots, r_n$. In case of a connected quandle Q, r_i and r_j are mutually conjugate for any pair i, j with $1 \leq i < j \leq n$ (Corollary 5.1 in [7]). This can be easily seen using Formula (6). Hence we call the pattern of r_n the profile of Q for short in this paper. For example, the profile of Q_{61} given in Table 1 is $\{1, 1, 2, 2\}$. In general, a non-trivial finite connected quandle is of profile $\{1, \ell_1, \ell_2, \cdots, \ell_k\}$ with $1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_k$ for some $k \in \mathbb{N}$. We have at least one 1 in profile since $r_i(i) = i$.

Conjecture 1.1. ℓ_k is a multiple of ℓ_i for any integer i with $1 \leq i \leq k-1$.

Lopes and Roseman studied finite quandles with profile $(\{1, n-1\}, \{1, n-1\}, \cdots, \{1, n-1\})$ in Theorem 6.5 and Corollaries 6.4–6.8 in [7]. They showed that the nth permutation r_n is $(1\ 2\ \cdots n-1)(n)$ modulo isomorphism, r_{n-1} is a solution to a certain system of equations, and r_k with $1 \leq k \leq n-2$ is determined by the formula $r_k = r_n^k r_{n-1} r_n^k$.

We define natural reorderings of the elements of a finite connected quandle.

Definition 1.2. Let Q be a connected quandle with n elements $1, 2, \cdots, n$. A bijection $\nu : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ is called a *natural reordering* (with respect to r_q) if $\nu(i_{st}) = (\sum_{j=1}^{s-1} \ell_j) + t$ and $\nu(q) = n$ for some element $q \in Q$ and some presentation of the permutation r_q as a product of disjoint cycles $r_q = (i_{11}\ i_{12}\ \cdots\ i_{1\ell_1})(i_{21}\ i_{22}\ \cdots\ i_{2\ell_2})\cdots(i_{k1}\ i_{k2}\ \cdots\ i_{k\ell_k})(q)$ with $1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_k$.

For example, for the quandle Q_{72} shown in Table 2, r_1 is decomposed to $(2\ 5\ 3)(4\ 6\ 7)(1)$. Hence $\lambda(1) = 7, \lambda(2) = 1, \lambda(3) = 3, \lambda(4) = 4, \lambda(5) = 2, \lambda(6) = 5$ and $\lambda(7) = 6$ give a

	1	2	3	4	5	6	7
1	1	5	2	6	3	7	4
2	5	2	6	3	7	4	1
3	2	6	3	7	4	1	5
4	6	3	7	4	1	5	2
5	3	7	4	1	5	2	6
6	7	4	1	5	2	6	3
7	4	1	5	2	6	3	7

TABLE 2. A quandle matrix for Q_{72}

reordering $\lambda : \{1, 2, \cdots, 7\} \to \{1, 2, \cdots, 7\}$ which is natural with respect to r_1. Since we can rewrite r_1 as $(4\ 6\ 7)(2\ 5\ 3)(1)$, $\mu(1) = 7, \mu(2) = 4, \mu(3) = 6, \mu(4) = 1, \mu(5) = 5, \mu(6) = 2$ and $\mu(7) = 3$ also give a natural reordering μ. The presentation $r_1 = (6\ 7\ 4)(3\ 2\ 5)(1)$ gives another natural reordering ν with $\nu(1) = 7, \nu(2) = 5, \nu(3) = 4, \nu(4) = 3, \nu(5) = 6, \nu(6) = 1$ and $\nu(7) = 2$. Since $r_2 = (1\ 5\ 7)(3\ 6\ 4)(2)$, $\xi(1) = 1, \xi(2) = 7, \xi(3) = 4, \xi(4) = 6, \xi(5) = 2, \xi(6) = 5$ and $\xi(7) = 3$ determine a natural reordering ξ with respect to r_2.

After the elements of Q are reordered naturally, the permutation r_n is decomposed into disjoint cycles in the form below, where $1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_k$. We call this form (N).

$$r_n = (1\ 2\ \cdots \ell_1)(\ell_1 + 1\ \ \ell_1 + 2\ \ \cdots\ \ \ell_1 + \ell_2)(\ell_1 + \ell_2 + 1\ \ \ell_1 + \ell_2 + 2\ \ \cdots\ \ \ell_1 + \ell_2 + \ell_3)$$
$$\cdots((\textstyle\sum_{j=1}^{k-1} \ell_j) + 1\ \ (\sum_{j=1}^{k-1} \ell_j) + 2\ \ \cdots\ \ (\sum_{j=1}^{k-1} \ell_j) + \ell_k)(n)$$

Note that there are $n\ell_1\ell_2 \cdots \ell_k$ natural reorderings if $1 \leq \ell_1 < \ell_2 < \cdots < \ell_k$. There are more if $\ell_i = \ell_{i+1}$ for some $1 \leq i \leq k - 1$.

Definition 1.3. Let Q be a finite connected quandle with n elements $1, 2, \cdots, n$. If r_n is decomposed into the form (N), then we say that the elements of Q are *naturally ordered*, and that the quandle matrix of Q is in *canonical form*.

Theorem 1.4. *Let Q be a finite connected quandle whose elements are naturally ordered. Then any automorphism of Q is a natural reordering.*

Theorem 1.5. *Let Q be a finite connected quandle with its elements $1, 2, \cdots, n$ naturally ordered, and with profile $\{1, \ell_1, \ell_2, \cdots, \ell_k\}$, where $1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_k$. For any reordering μ which is natural with respect to r_q for some $q \in Q$, there is a natural reordering ν with respect to r_n such that the quandle matrices after reordering by μ and ν coincide. Moreover, we can take ν so that, for some integer m with $1 \leq m \leq k$ and $\ell_m = \ell_k$, $\nu((\sum_{j=1}^{m-1} \ell_j) + i) = (\sum_{j=1}^{k-1} \ell_j) + i$ for any integer i with $1 \leq j \leq \ell_m$. In particular, when $\ell_{k-1} < \ell_k$, we can take ν so that $\nu(i) = i$ for any integer i with $(\sum_{j=1}^{k-1} \ell_j) + 1 \leq i \leq n$.*

Conjecture 1.6. In the last sentence of Theorem 1.5, we can take ν so that $\nu(i) = i$ for any integer i with $(\sum_{j=1}^{k-1} \ell_j) + 1 \leq i \leq n$ even when $\ell_{k-1} = \ell_k$.

	1	2	3	4	5
1	1	4	5	3	2
2	4	2	1	5	3
3	5	1	3	2	4
4	3	5	2	4	1
5	2	3	4	1	5

TABLE 3. The canonical quandle matrix for Q_{52}

	1	2	3	4	5
1	1	5	4	3	2
2	4	2	5	1	3
3	2	1	3	5	4
4	5	3	2	4	1
5	3	4	1	2	5

TABLE 4. The canonical quandle matrix for Q_{53}

order	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
quandles	1		1	1	2,3		4,5	2,3	7,8		6-9		1,5,6,10		

order	16	17	18	19	20	21	22	23	24
quandles	8,9	2,4-6,9-11,13		1,2,9,12-14				4,6,9,10,13,14,16,18-20	

order	25	26	27	28	29	30	31
quandles	31-34		62-65		1,2,7,9,10,13,14,17,18,20,25,26		2,10-12,16,20,21,23

order	32	33	34	35
quandles	10-15			

TABLE 5. Connected quandles with profile $\{1, n-1\}$

Corollary 1.7. *Let Q be a finite connected quandle with its elements naturally ordered, and with profile $\{1, \ell_1, \ell_2, \cdots, \ell_k\}$, where $1 \leq \ell_1 < \ell_2 < \cdots < \ell_k$. Then the number of canonical forms of quandle matrices for Q after reorderings are at most $\ell_1 \ell_2 \cdots \ell_{k-1}$. In particular, when $k = 1$ or "$k = 2$ and $\ell_1 = 1$", the canonical form of quandle matrix for Q is unique.*

The canonical forms of the quandle matrices for Q_{52} and Q_{53} in Vendramin's list are shown in Tables 3 and 4. They are of profile $\{1, 4\}$. We can see that Q_{52} and Q_{53} are not isomorphic because the canonical forms of their quandle matrices are distinct.

The next corollary immediately follows from Corollary 5 in [4], Theorem 1.4 and Corollary 1.7.

Corollary 1.8. *Let Q be a finite connected quandle of order n with its elements naturally ordered, and with profile $\{1, n-1\}$ or $\{1, 1, n-2\}$. Then the set of all the automorphisms of Q coincides with the set of all the natural reorderings of Q.*

There are many quandles as in the above corollary. The connected quandles with profile $\{1, n-1\}$ in Vendramin's list are shown in Table 5. For example, the connected quandles with 19 elements with profile $\{1, 18\}$ are $Q_{19,1}$, $Q_{19,2}$, $Q_{19,9}$, $Q_{19,12}$, $Q_{19,13}$ and $Q_{19,14}$. The quandle Q_{62} in Vendramin's list has profile $\{1, 1, 4\}$.

Let Q be a finite quandle. For an element $i \in Q$, the map $l_i : Q \ni x \mapsto i * x \in Q$ is not necessarily a bijection in general. In [4], Ho and Nelson defined a *latin* quandle to be a quandle with the map l_i being bijection for any $i \in Q$. They showed that any conjugation quandle of a group is latin. The quandles Q_{31}, Q_{41}, Q_{51}, Q_{52} and Q_{53} in Vendramin's list are latin. Q_{61} shown in Table 1 is the first example of a connected non-latin quandle. The next theorem is a generalization of Corollary 6.4 in [7], and is due to Kanako Oshiro. The converse is not true since Q_{51} in Vendramin's list is latin and of profile $\{1, 2, 2\}$.

Theorem 1.9. (K. Oshiro) *Let Q be a connected quandle with n elements. If the profile of Q is $\{1, n-1\}$, then Q is latin.*

We prove Theorems 1.4 and 1.5 in the next section, and Theorem 1.9 in Section 3.

2. Proofs of Theorems 1.4 and 1.5

Throughout this section, let Q be a finite connected quandle with n elements $1, 2, \cdots, n$ and with profile $\{1, \ell_1, \ell_2, \cdots, \ell_k\}$, where $1 \le \ell_1 \le \ell_2 \le \cdots \le \ell_k$.

Lemma 2.1. *A bijection $\nu : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ is a natural reordering with respect to $r_{\nu^{-1}(n)}$ if and only if the new quandle matrix after reordering by ν is canonical.*

Proof. The "only if part" is very clear. We show the "if part". Let r_b and $R_b : Q \ni x \mapsto x * b \in Q$ be the permutations before and after the reordering by ν respectively. Since the new quandle matrix is canonical, the permutation R_n is decomposed in the form (N) shown in Introduction. Set $\nu^{-1}(n) = q$, and $\nu^{-1}((\sum_{j=1}^{s-1} \ell_j) + t) = i_{st}$ for any pair of integers s and t with $1 \le s \le k$ and $1 \le t \le \ell_s$. Then $r_q(i_{st}) = i_{st} * q = \nu^{-1}((\sum_{j=1}^{s-1} \ell_j) + t) * \nu^{-1}(n) = \nu^{-1}(((\sum_{j=1}^{s-1} \ell_j) + t) * n) = \nu^{-1}(R_n(((\sum_{j=1}^{s-1} \ell_j) + t))) = \nu^{-1}((\sum_{j=1}^{s-1} \ell_j) + (t+1)) = i_{s\,t+1}$ where $t+1$ is read to be an integer in the interval $[1, \ell_s]$ modulo ℓ_s. We can see that the third equality holds from the way of construction of the new quandle matrix after reordering shown in Introduction (see Definition 1 in [4]). Hence, ν is the natural reordering with respect to the presentation $r_q = (i_{11}\ i_{12}\ \cdots\ i_{1\ell_1}) \cdots (i_{k1}\ i_{k2}\ \cdots\ i_{k\ell_k})(q)$. $\square$

Lemma 2.2. *For any element q in Q, there is an automorphism ν of Q such that $\nu(q) = n$.*

Proof. Since Q is connected, there is a set of elements $i_1, i_2, \cdots, i_m$ of Q such that $((((q \star_1 i_1) \star_2 i_2) \star_3 \cdots) \star_m i_m) = n$, where $\star_j \in \{*, \bar{*}\}$ for any integer j with $1 \le j \le m$. Then the composition $\nu = r_{i_m}^{\epsilon_m} \circ r_{i_{m-1}}^{\epsilon_{m-1}} \circ \cdots \circ r_{i_2}^{\epsilon_2} \circ r_{i_1}^{\epsilon_1}$ with $\epsilon_j = +1$ (when $\star_j = *$) and -1 (when $\star_j = \bar{*}$) brings q to n. Note that ν is an automorphism of Q because $r_{i_j}^{\pm 1}$ is an automorphism of Q for each j with $1 \le j \le m$. $\square$

In the rest of this section, we assume that the elements of Q are naturally ordered, and hence the quandle matrix is in canonical form. Under this condition, we can easily show the next three lemmas. We omit the proofs.

Lemma 2.3. *Let μ be a natural reordering with respect to r_q for some $q \in Q$, and ν a natural reordering with respect to r_n. Then the composition $\nu \circ \mu$ is a natural reordering with respect to r_q.*

Lemma 2.4. *Let ν be a natural reordering with respect to r_n. Then the inverse map ν^{-1} is also a natural reordering with respect to r_n.*

Lemma 2.5. *The set of all the natural reoderings with respect to r_n forms a subgroup of the symmetric group on $\{1, 2, \cdots, n\}$.*

Proof of Theorem 1.4. If μ is an automorphism of Q, then it fixes the quandle matrix by Corollary 5 in [4]. In particular, μ unchanges r_n, and hence the quandle matrix is canonical also after reordering by μ. Thus μ is a natural reordering by Lemma 2.1. □

Proof of Theorem 1.5. Because μ is natural with respect to r_q, we have $\mu(q) = n$. Since Q is connected, there is an automorphism λ of Q with $\lambda(q) = n$ by Lemma 2.2. Then the reordering $\nu = \mu \circ \lambda^{-1}$ fixes n. Since λ is an automorphism of Q, it fixes the quandle matrix (Corollary 5 in [4]). Hence the new quandle matrices after reordering by ν and μ coincide. Because the natural reordering μ fixes r_n in the form (N) in Introduction, also ν does, and hence ν is a natural reordering with respect to r_n by Lemma 2.1.

Moreover, since ν is natural with respect to r_n, $\nu^{-1}((\sum_{j=1}^{k-1} \ell_j) + i) = (\sum_{j=1}^{m-1} \ell_j) + t + i$ holds for some integers m, t with $1 \leq m \leq k$, $0 \leq t \leq \ell_m - 1$ and $\ell_m = \ell_k$ and for any integer i with $1 \leq i \leq \ell_m$, where $t + i$ is read to be an integer in the interval $[1, \ell_m]$ modulo ℓ_m. Then we consider the reordering $\nu' = r_n^t \circ \nu = r_n^t \circ (\mu \circ \lambda^{-1})$. Note that $\nu'(n) = n$ and $\nu'((\sum_{j=1}^{m-1} \ell_j) + i) = ((\sum_{j=1}^{k-1} \ell_j) - t + i) + t = (\sum_{j=1}^{k-1} \ell_j) + i$ for any integer i with $1 \leq i \leq \ell_m$. The isomorphism r_n^t does not change the quandle matrix. Hence the quandle matrices after reordering by μ and ν' coincide, and ν' is natural with respect to r_n by Lemma 2.1. □

3. Connected quandle with profile $\{1, n-1\}$

Let Q be a finite connected quandle of order n and with profile $\{1, n-1\}$.

Proof of Theorem 1.9. Suppose, for a contradiction, that Q is not latin. Then $k * i = k * j$ for some elements $i, j, k \in Q$ with $i \neq j$.

Suppose first that $k = i$ or $k = j$, say $k = i$. Then $i * j = k * j = k * i = i * i = i$. Hence r_j fixes j and i. Since Q is of profile $\{1, n-1\}$, we have $n - 1 = 1$, and hence $n = 2$, which contradicts the fact that there is no connected quandle of order 2.

Then we can assume that $k \neq i$ and $k \neq j$. Note that r_k is a cycle of length $n - 1$ and fixes k, and hence r_k is a cycle on $n - 1$ letters containing i and j. Then, there is an integer m with $1 \leq m \leq n - 2$ and $r_k^m(i) = j$. Hence, by Formula (6) in Introduction, $k * i = k * j = k * (r_k^m(i)) = r_k^m(r_i(r_k^{-m}(k))) = r_k^m(r_i(k)) = r_k^m(k * i)$. This means that r_k^m

fixes $k * i$. Because $1 \leq m \leq n-2$, the cycle r_k fixes $k * i$, and hence $k = k * i$. Then r_i fixes i and k, which is a contradiction. $\square$

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